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A Finite Crisscross Method for Oriented Matroids

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Our paper presents a new finite crisscross method for oriented matroids. Starting from a neither primal nor dual feasible tableau, we reach primal and dual optimal oriented circuits in a finite number of steps if they exist. If there is no optimal tableau then we show that there is no primal feasible circuit or there is no dual feasible cocircuit. So we give a new constructive proof for the general duality theorem (Bland *J. Combin. Theory Ser. B* **23** (1977), 33–57; Folkman and Lawrence *J. Combin. Theory Ser. B* **25** (1978), 199–236). Our pivot rule is a generalization of the "anticycling rule" suggested in Bland (op cit; *Math. Oper. Res.* **2** (1977), 103–107). Finite pivoting rules are given by Edmonds, Fukuda and Todd (Ph.D. dissertation, Univ. of Waterloo, 1982), *SIAM Algebraic Discrete Math.* **5**, No. 4 (1984), 467–485). A general relaxed recursive algorithm was discovered independently by Jensen (Ph.D. thesis, School of OR and IE, Cornell, 1985) which is principally crisscross type. Jensen's is very general and flexible; in fact it can be considered as a family of algorithms. Among the conceivable algorithms in his general family our independently constructed crisscross method is characterized by its extreme simplicity. © 1987 Academic Press, Inc.

1. INTRODUCTION

Zionts [15] in 1969 presented his crisscross method for linear programs. Until now, it is an open question whether Zionts' crisscross method is finite or not, even in the case of linear programming. In the theory of linear programming another interesting idea was the finite pivoting rule of Bland [3], by which cycling can be avoided in the case of degeneracy. The investigation and the educational usage of these two results and the use of tableaus in presenting feasibility and duality theorems by Balinski and Tucker [1], resulted in the finite crisscross method in the end of 1983. This crisscross method is presented in Terlaky [10].

The first step was made by Rockafellar [8] towards the combinatorial abstraction of linear programming. Using Rockafellar's results, Bland [2], Bland and Las Vergnas [4], and Folkman and Lawrence [5] established

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the theory of oriented matroids. It was shown by Bland [2], that a dual pair of linear programs gives a dual pair of oriented matroids. Bland [2] and Folkman and Lawrence [5] proved the general duality theorem (Theorem 3.5 in [2]). Here we give a new constructive proof for this theorem by a simple, finite pivoting rule.

Since only the sign properties (0, +, − structure) of the basic tableau are used by the finite crisscross method even in the case of linear programming, it was a natural step to investigate how we can generalize the crisscross method for oriented matroids. We could perform this generalization by using the tableau construction and the equivalent axiomatization of oriented matroids published by Bland and Las Vergnas [2, 4].

Our crisscross method produces a simple constructive proof for the general duality theorem Bland [2] and Folkman and Lawrence [5]. Different finite pivoting rules were given by Edmonds, Fukuda [6], and Todd [12]. A general relaxed recursive algorithm was discovered independently by Jensen [6] in his thesis, which is principally crisscross type. Jensen's algorithm is very general and flexible; in fact it can be considered as a family of algorithms. Among the conceivable algorithms in his general family our independently constructed crisscross method is characterized by its extreme simplicity. An anonymous referee kindly called my attention to the independent work of Wang [17] who recently constructed a "finite conformal-elimination-free algorithm," based on ideas analogous to ours.

In this paper we use the equivalent axiomatizations of oriented matroids presented by Bland and Las Vergnas [4] and we use their notations. We will use the basic tableau construction of Bland [2] (which is nearly the same as in linear programming) and the properties of these tableaus.

For completeness we recall the definition of oriented matroids and the tableau construction.

Let $E = \{e_1, ..., e_n\}$ be a finite set. A pair $X = (X^+, X^-)$ is called a *signed set* if $X^+, X^- \subset E$ and $X^+ \cap X^- = \emptyset$. We use the following notations: $\mathbf{X} = X^+ \cup X^-$, $-X = ((-X)^+, (-X)^-) = (X^-, X^+)$.

DEFINITION 1.1. Let θ and θ^* be collections of signed sets of E. The pairs $M = (E, \theta)$ and $M^* = (E, \theta^*)$ are called *dual pairs of oriented matroids* if the following four conditions are satisfied.

(a) $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$ are the circuits and cocircuits of a dual pair of matroids $\mathbf{M} = (E, \boldsymbol{\theta})$, $\mathbf{M} = (E, \boldsymbol{\theta}^*)$.

(b) $X \in \theta \Rightarrow -X \in \theta$; $Y \in \theta^* \Rightarrow -Y \in \theta^*$.

(c) $X_1, X_2 \in \theta$ and $\mathbf{X}_1 = \mathbf{X}_2 \Rightarrow X_1 = \pm X_2$; $Y_1, Y_2 \in \theta^*$ and $\mathbf{Y}_1 = \mathbf{Y}_2 \Rightarrow Y_1 = \pm Y_2$.

(d) If $X \in \theta$, $Y \in \theta^*$ and $\mathbf{X} \cap \mathbf{Y} \neq \varnothing$ then $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^- \cap Y^+) \cup (Y^- \cap X^+) \neq \varnothing$.

Assumption (d) is called the *orthogonality* condition.

Let $B = \{b_1, ..., b_m\}$ be a base of the oriented matroid $M = (E, \theta)$. A tableau $T(B)$ is a matrix which contains the signed incidence vectors of the oriented cocircuits associated to the basic elements b_i, $i = 1, ..., m$. In order to ease the description, the row of $T(B)$ associated with the oriented cocircuit Y_{b_i} is called the b_ith row of $T(B)$ (not the ith row as it is usual in natural ordering). So the entry t_{ij} of $T(B)$ is the sign of e_j in Y_i, where $Y_i \in \{Y_{b_1}, ..., Y_{b_m}\}$. The detailed construction and the properties of the tableau $T(B)$ can be found in Bland [2].

The operation of replacing $T(B)$ by $T(\bar{B})$ so that $\bar{B} = (B \cup \{e_k\}) \setminus \{e_j\}$ ($e_j \in B$, $e_k \notin B$, $t_{jk} \neq 0$) is called a *pivot* on position (j, k). (It is well known that $\bar{B}$ is a base again in this case.)

2. The Finite Crisscross Method and the Proof of the Duality Theorem

Let $M = (E, \theta)$, $M^* = (E, \theta^*)$ be dual pairs of oriented matroids, where $E = \{e_1, ..., e_n\}$.

Definition 2.1. The oriented circuit $X \in \theta$ is called *primal feasible* if $e_1 \in X^+$, $X^- \subset \{e_2\}$.

Definition 2.2. The oriented cocircuit $Y \in \theta^*$ is called *dual feasible* if $e_2 \in Y^+$, $Y^- \subset \{e_1\}$.

Remark. The previously defined sets are called extremal or basic feasible.

Definition 2.3. The signed sets X and Y are called *complementary* sets if $\mathbf{X} \cap \mathbf{Y} \subset \{e_1, e_2\}$.

Definition 2.4. An $X \in \theta$ primal feasible oriented circuit is called *primal optimal* if there is a $Y \in \theta^*$ dual feasible oriented cocircuit such that X and Y are complementary sets.

Definition 2.5. A $Y \in \theta^*$ dual feasible oriented cocircuit is called *dual optimal* if there is an $X \in \theta$ primal feasible oriented circuit such that X and Y are complementary sets.

The problem we investigate in this section is the following.

PROBLEM I. Find a primal optimal oriented circuit $X \in \theta$ and a dual optimal oriented cocircuit $Y \in \theta^*$ or show that there is no optimal oriented circuit or cocircuit.

We assume in our further considerations that $e_1 \notin B$ and $e_2 \in B$. The pivoting rule will preserve this property. The e_1 column of $T(B)$ corresponds to an oriented circuit X_1 and the e_2 row of $T(B)$ corresponds to an oriented cocircuit Y_2 in each tableau $T(B)$.

Keeping the linear programming terminology we will refer to the e_2 row of $T(B)$ as the *objective function row* and we will refer to the e_1 column as the *solution column*.

We say that a tableau $T(B)$ is optimal if $(-X_1)$ and Y_2 are primal and dual feasible, respectively, since in this case $(-X_1)$ and Y_2 are complementary sets; that is, they are optimal.

If $\{e_2\} \in \theta$ then there is no tableau since in this case there is no such $B \in \mathscr{B}$ that $e_2 \in B$. So we assume that $\{e_2\} \notin \theta$. The details of this case can be found in Bland [2]. The further proofs and the crisscross method are new.

The following two lemmas show how we can see from a tableau that there is no primal or dual feasible circuit.

LEMMA 2.6. *If for an $e_k \notin B$, $k \neq 1$, $t_{2k} = -1$, and $t_{ik} \in \{-1, 0\}$, $e_i \in B$, then there is no dual feasible oriented cocircuit $Y \in \theta^*$.*

Proof. Let us suppose to the contrary that there is a dual feasible oriented cocircuit $Y \in \theta^*$, that is, $Y = (Y^+, Y^-)$, $e_2 \in Y^+$, $Y^- \subset \{e_1\}$. Consider the oriented circuit X_k which is associated to the e_k column of $T(B)$. So $e_2 \in \mathbf{X}_k \cap \mathbf{Y} \neq \varnothing$, and $\{e_2\} \subset (X_k^+ \cap Y^-) \cup (X_k^- \cap Y^+) \neq \varnothing$ but $(X_k^+ \cap Y^+) \cup (X_k^- \cap Y^-) = \varnothing$ since $X_k^+ = \varnothing$, $e_1 \notin X_k^-$, and $Y^- \subset \{e_1\}$. This is a contradiction; our proof is complete.

LEMMA 2.7. *If for an $e_k \in B, k \neq 2$, $t_{k1} = +1$, and $t_{ki} \in \{0, +1\}$, $e_i \notin B$, then there is no primal feasible oriented circuit $X \in \theta$.*

Proof. This lemma is the dual of Lemma 2.6.

These two lemmas are the same as Lemma 3.4.4 in Bland [2]. We have proved them again in order to preserve the unity of our paper.

If a base B and the tableau $T(B)$ are given, the pivoting rule which defines the crisscross method is the following.

PIVOTING RULE I. (a) If $t_{2j} \in \{0, +1\}$, $e_j \notin B, j \neq 1$, and $t_{i1} \in \{-1, 0\}$ $e_i \in B$ $i \neq 2$ then the oriented circuit $(-X_1)$ is primal feasible and the oriented cocircuit Y_2 is dual feasible, that is, $T(B)$ is an optimal tableau. The algorithm is completed; Problem I is solved.

(b) If the case (a) does not hold, then denote

$$k=\min\{i: t_{2i}=-1 \quad \text{or} \quad t_{i1}=+1, i>2\}.$$

(c) (i) If $t_{2k}=-1$ and $t_{ik}\in\{-1,0\}$, $e_i\in B$, then by Lemma 2.6 there is no dual feasible oriented cocircuit $Y\in\theta^*$. The algorithm is completed; Problem I is solved.

(ii) *Primal transformation.* If $t_{2k}=-1$ and $t_{ik}=+1$ for an $e_i\in B$ then e_k enters the base. Denote $r=\min\{i: t_{ik}=+1, i>2\}$. The element e_r leaves the base. Make a pivot operation on the position (r, k). $\bar{B}=(B\cup\{e_k\})\setminus\{e_r\}$.

(d) (i) If $t_{k1}=+1$ and $t_{ki}\in\{0,+1\}$, $e_i\notin B$, then by Lemma 2.7 there is no primal feasible oriented circuit $X\in\theta$. The algorithm is completed; Problem I is solved.

(ii) *Dual transformation.* If $t_{k1}=+1$ and $t_{ki}=-1$ for an $e_i\notin B$ then e_k leaves the base. Denote $s=\min\{i: t_{ki}=-1, i>2\}$. The element e_s enters the base. Make a pivot operation on the position (k, s). $\bar{B}=(B\cup\{e_s\})\setminus\{e_k\}$.

We continue our procedure with the new base $\bar{B}$. Since the pivot element is not zero, so $\bar{B}$ is also a base.

Our procedure stops at one of the cases (a), (c)(i), or (d)(i). In case (a) we have an optimal tableau, at the cases (c)(i) or (d)(i) there is no dual or primal feasible circuit. To solve Problem I, one has to prove only that the crisscross method defined by Pivoting Rule I cannot produce a cycle of pivots, that is, a $B\in\mathscr{B}$ base may occur at most once, if we use Pivoting Rule I.

THEOREM 2.8. *The crisscross method cannot produce any cycle; that is, our procedure stops after a finite number of steps.*

Proof. Let us suppose to the contrary that cycling occurs through the procedure; that is, starting from a base B we get again the base B. Let $E^c=\{e_i: e_i$ leaves the base through the cycle$\}$. We note that $e_i\notin E^c$ implies that e_i was a basic or a nonbasic element through the entire cycle. Denote $q=\max\{i: e_i\in E^c\}$.

Let us consider those two situations when e_q enters and e_q leaves the base. Let B' and B'' be the two bases respectively, and distingwish by $'$ and $''$ the components of the tableaus $T(B')$ and $T(B'')$. Let e_r be the element leaving the base when e_q enters and let e_s be the element entering the base when e_q leaves the base. It is obvious that $q>2$, $r, s<q$, and $e_r, e_s\in E^c$.

One has to consider the following four cases:

(α) e_q enters and leaves the base at primal transformations.

(β) e_q enters the base at a primal transformation and leaves the base at a dual transformation.

(γ) e_q enters the base at a dual transformation and leaves the base at a primal transformation.

(δ) e_q enters and leaves the base at dual transformations.

Let us examine these four cases. We shall see that all the four cases lead to a contradiction; that is, cycling cannot occur.

(α) The element e_q enters and e_r leaves the base at primal transformation with base B'. The element e_s enters and e_q leaves the base at primal transformation with base B''. By Pivoting Rule I, $Y'_2 \in \theta^*$ and $X''_s \in \theta$ have the following properties:

(1′) $e_q \in Y_2'^-$	(1″) $e_q \in X_s''^+$
(2′) $e_2 \in Y_2'^+$	(2″) $e_2 \in X_s''^-$
(3′) $Y_2'^- \cap E^c = \{e_q\}$	(3″) $X_s''^+ \cap E^c = \{e_q\}$
(4′) $\mathbf{Y}'_2 \subset (E - B') \cup \{e_2\}$	(4″) $\mathbf{X}''_s \subset B'' \cup \{e_s\}$

Properties (1′), (1″) imply that $e_q \in \mathbf{X}''_s \cap \mathbf{Y}'_2 \neq \emptyset$. Using (4′), (4″) we have $\mathbf{X}''_s \cap \mathbf{Y}'_2 \subset [B'' \cup \{e_s\}] \cap [(E - B') \cup \{e_2\}] \subset E^c \cup \{e_2\}$, and so (3′), (3″) implies that $(X_s''^+ \cap Y_2'^+)$, $((X_s''^- \cap Y_2'^-) \subset \{e_2, e_q\}$. Using properties (1′), (1″), (2′), (2″) we have that $X_s''^+ \cap Y_2'^+ = X_s''^- \cap Y_2'^- = \emptyset$, which contradicts the orthogonality of X''_s and Y'_2; so this case is impossible.

(β) The element e_q enters and e_r leaves the base at primal transformation with base B'. The element e_s enters and e_q leaves the base at dual transformation with base B''. Let us consider the circuits X'_1, X''_1 and the cocircuits Y'_2, Y''_2.

Bland and Las Vergnas in [4, Theorem 2.2] have proved that the orthogonality condition is equivalent to the following assumption:

For all $X_1, X_2 \in \theta$, $e' \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, and $e'' \in (X_1^+ - X_2^-) \cup (X_1^- - X_2^+)$, there exists $X_3 \in \theta$ such that $X_3^+ \subset (X_1^+ \cup X_2^+) - \{e'\}$, $X_3^- \subset (X_1^- \cup X_2^-) - \{e'\}$, and $e'' \in \mathbf{X}_3$.

If we use this condition with $X_1 = X''_1, X_2 = -X'_1, e' = e_1$, $e'' = e_q$, then we have a circuit $X \in \theta$ which by Pivoting Rule I has the following properties:

(1′) $e_1 \notin \mathbf{X}$

(2′) $e_q \in X^+$

(3′) $X^+ \subset \{e_q\} \cup B' \cup (B'' - E^c)$

(4′) $X^- \subset (B'' - \{e_q\}) \cup (B' - E^c)$.

Using again the above presented condition with $(-Y'_2)$, Y''_2, $e' = e_2$,

$e'' = e_q$, then we have a cocircuit $Y \in \theta^*$, which by Pivoting Rule I has the following properties:

(1″) $e_2 \notin \mathbf{Y}$

(2″) $e_q \in Y^+$

(3″) $Y^+ \subset \{e_q\} \cup [(E-B') - E^c] \cup (E-B'')$

(4″) $Y^- \subset [(E-B') - \{e_q\}] \cup [(E-B'') - E^c]$

The properties (2′), (2″) imply that $e_q \in \mathbf{X} \cap \mathbf{Y} \neq \varnothing$, but the properties (2′), (3′), (2″), (4″) imply that $X^+ \cap Y^- \subset \{\{e_q\} \cup B' \cup (B'' - E^c)\} \cap \{[(E-B') - \{e_q\}] \cup [(E-B'') - E^c]\} = \varnothing$, and the properties (2′), (4′), (2″), (3″) imply that $X^- \cap Y^+ \subset \{(B'' - \{e_q\}) \cup (B' - \mathrm{E}^c)\} \cap \{\{e_q\} \cup [(E-B') - E^c] \cup (E-B'')\} = \varnothing$. This contradicts the orthogonality of X and Y, so this case is also impossible.

(γ) The element e_q enters and e_r leaves the base at dual transformation with base B'. The element e_s enters and e_q leaves the base at primal transformation with base B''. By Pivoting Rule I, $Y'_r \in \theta^*$ and $X''_s \in \theta$ have the following properties:

(1′) $e_1 \in Y_r'^+$	(1″) $e_2 \in X_s''^-$
(2′) $e_q \in Y_r'^-$	(2″) $e_q \in X_s''^+$
(3′) $Y_r'^- \cap E^c = \{e_q\}$	(3″) $X_s''^+ \cap E^c = \{e_q\}$
(4′) $\mathbf{Y}'_r \subset (E-B') \cup \{e_r\}$	(4″) $\mathbf{X}''_s \subset B'' \cup \{e_s\}$.

The properties (2′), (2″) imply that $e_q \in \mathbf{X}''_s \cap \mathbf{Y}'_r \neq \varnothing$. Using (4′), (4″) and the definition of E^c we have $\mathbf{X}''_s \cap \mathbf{Y}'_r \subset [B'' \cup \{e_s\}] \cap [(E-B') \cup \{e_r\}] \subset E^c$. So (3′), (3″) imply that $Y_r'^+ \cap X_s''^+ \subset \{e_q\}$ and $Y_r'^- \cap X_s''^- \subset \{e_q\}$, and using the properties (2′), (2″) we have $Y_r'^+ \cap X_s''^+ = \varnothing$ and $Y_r'^- \cap X_s''^- = \varnothing$, which contradicts the orthogonality of Y'_r and X''_s. This case is also impossible.

(δ) The element e_q enters and e_r leaves the base at dual transformation with base B'. The element e_s enters and e_q leaves the base at dual transformation with base B''. By Pivoting Rule I, $Y'_r \in \theta^*$ and $X''_1 \in \theta$ have the following properties:

(1′) $e_q \in Y_r'^-$	(1″) $e_q \in X_1''^+$
(2′) $e_1 \in Y_r'^+$	(2″) $e_1 \in X_1''^-$
(3′) $Y_r'^- \cap E^c = \{e_q\}$	(3″) $X_1''^+ \cap E^c = \{e_q\}$
(4′) $\mathbf{Y}'_r \subset (E-B') \cup \{e_r\}$	(4″) $\mathbf{X}''_1 \subset B'' \cup \{e_1\}$.

The properties (1′), (1″) imply that $e_q \in \mathbf{X}''_1 \cap \mathbf{Y}'_r \neq \varnothing$. Using (4′), (4″) and the definition of E^c we have $\mathbf{X}''_1 \cap \mathbf{Y}'_r \subset [B'' \cup \{e_1\}] \cap$

$[(E-B')\cup\{e_r\}]\subset E^c\cup\{e_1\}$. So (3′), (3″) imply that $Y_r'^{+}\cap X_1''^{+}\subset\{e_1,e_q\}$ and $Y_r'^{-}\cap X_1''^{-}\subset\{e_1,e_q\}$ and using the properties (1′), (1″), (2′), (2″) we have that $Y_r'^{+}\cap X_1''^{+}=\emptyset$ and $Y_r'^{-}\cap X_1''^{-}=\emptyset$. This contradicts the orthogonality of X_1'' and Y_r'; that is, this case is also impossible.

Since all four cases led to a contradiction, we have proved that cycling cannot occur; our procedure is finite. The proof is complete.

Remark. Using duality properties of oriented matroids we could have left case (δ), since it is the dual of case (α). Unfortunately we can not leave case (γ), since cases (β) and (γ) are self duals; so we have to consider at least three cases to prove our theorem.

So we have a new algorithmic proof for the general duality theorem of Bland [2], Folkman and Lawrence [4].

THEOREM 2.9. *Let $M=(E,\theta)$ and $M^*=(E,\theta^*)$ be dual pairs of oriented matroids. Exactly one of the following alternatives* (a) *and* (b) *holds:*

(a) *There is an $X\in\theta$ such that $e_1\notin \mathbf{X}$, $e_2\in X^+$, and $X^-=\emptyset$ or there is a $Y\in\theta^*$ such that $e_1\in Y^+$, $e_2\notin\mathbf{Y}$, and $Y^-=\emptyset$; or*

(b) *There is an $X\in\theta$, $Y\in\theta^*$ such that $e_1\in X^+$, $X^-\subset\{e_2\}$,*

$$e_2\in Y^+,\ Y^-\subset\{e_1\}\qquad \textit{and}\qquad \mathbf{X}\cap\mathbf{Y}\subset\{e_1,e_2\}.$$

Proof. The orthogonality of oriented circuits and cocircuits imply that both of (a) and (b) cannot hold simultaneously.

If $\{e_2\}\in\theta$, then alternative (a) holds, and alternative (b) fails obviously.

If $\{e_2\}\notin\theta$ then the three outcomes of the crisscross method give the proof of this theorem, since Theorem 2.8 proves the finiteness of the crisscross method. The proof is complete.

3. REMARKS ON THE METHOD

We can solve the feasibility problem by the finite crisscross method too. The feasibility problem is the following.

DEFINITION 3.1. The oriented circuit is called *feasible*, if $e_1\in X^+$, $X^-=\emptyset$.

The problem is to find a feasible oriented circuit if one exists, or show that there is no feasible oriented circuit in M.

If the objective function element (e_2) is an oriented cocircuit then we have the "one-sided" version of the crisscross method. In this special case our method is the same as Bland's [2] least subscript rule, since $t_{2i}=0$, $i=3,\dots,n$, for all tableau. In this case every pivot is degenerate, so Bland's method is finite.

So this special algorithm and the algorithmic proof of the Farkas lemma and Minty's painting lemma (Bland [2, Lemma 3.1]) are immediate consequences of the crisscross method.

The linear programming version of this method is presented in Terlaky [10]. Recently Roos [9] proved that this method is exponential even in the case of linear programming.

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